ANSWERS TO SOME OF THE EE 202 HOMEWORK PROBLEMS

HW-6. Solve $\frac{dy}{dx} = \sin(x+y) - e^x$ and $y(0) = 4$ from x=0 to x=0.2 with steps h=0.1 by using Euler's method.

Answer:

Initial value problem is $\frac{dy}{dx} = \sin(x+y) - e^x$ and $y(0) = 4$

Step 1: Start at the point $(x_0, y_0) = (0,4)$ and use step size h=0.1 and use 2 steps, i.e., the solution of the differential equation will be approximated from x=0 to x=0.2.

Calculating the value of the derivative $\frac{dy}{dx} = f(x,y)$ at the initial point $x_0 = 0, y_0 = 4$

$$\frac{dy}{dx} = \sin(x+y) - e^x = f(0,4) = \sin(0+4) - e^0 = \sin(4) - e^0 = -0.7568 - 1 = -1.7568$$

i.e., slope of the line from x=0 to x=0.1 is approximately -1.7568

Step 2: Next point is x+h=0+0.1=0.1

Substituting in the Euler Method's formula $y(x+h) \approx y(x) + hf(x,y)$

$$y_1 = y(x_0 + h) \approx y(x_0) + hf(x_0, y_0) \Rightarrow y_1 = y(x_0 + h) \approx y_0 + hf(x_0, y_0)$$

$$y_1 = y(0+0.1) \approx y_0 + 0.1\left[\sin(x_0 + y_0) - e^{x_0}\right] = 4 + 0.1\left[\sin(0+4) - e^0\right] = 4 + 0.1(-1.7568) = 3.82432$$

i.e., the approximate value of the solution at x=0.1 is 3.82432.

Now, we find the new slope at $x_1 = 0.1, y_1 = 3.82432$

$$\frac{dy}{dx} = \sin(x+y) - e^x = f(0.1, 3.82432) = \sin(0.1 + 3.82432) - e^{0.1} = -1.81039$$

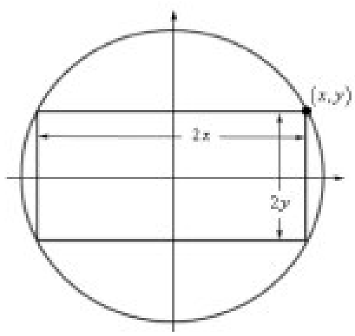
i.e., slope of the line from x=0.1 to x=0.2 is approximately -1.81039

Step 3: We find the solution value when x=0.2.

$$y_2 = y(x_1 + h) = y(2.2) \approx y(x_1) + hf(x_1, y_1) \Rightarrow y_2 = y(x_1 + h) \approx y_1 + hf(x_1, y_1)$$
$$= 3.82432 + 0.1 \times f(0.1, 3.82432)$$
$$= 3.82432 + 0.1 \times \left[\sin(0.1 + 3.82432) - e^{0.1}\right] = 3.643281$$

Then the third iteration starts.

HW-11. Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.



Answer: Maximize $A = 4xy$

Constraint: $x^2 + y^2 = 4^2 = 16$

Solving the second equation for y

$$x^2 + y^2 = 16 \Rightarrow y = \sqrt{16 - x^2}$$

Substituting the result into the first equation

$$A = 4xy \Rightarrow A = 4x\sqrt{16 - x^2}$$

To find the absolute maximum value of $A = 4x\sqrt{16 - x^2}$,

The derivative of $A(x)$ is

$$A'(x) = \left(4x\sqrt{16 - x^2}\right)' = 4x\frac{1}{2}\left(16 - x^2\right)^{-1/2}(-2x) + 4\sqrt{16 - x^2} = -4x^2\left(16 - x^2\right)^{-1/2} + 4\sqrt{16 - x^2}$$

$$= -4x^2\left(16 - x^2\right)^{-1/2} + 4\sqrt{16 - x^2} = -\frac{4x^2}{\sqrt{16 - x^2}} + 4\sqrt{16 - x^2} = \frac{-4x^2 + 4\left(16 - x^2\right)}{\sqrt{16 - x^2}}$$

To find the critical numbers we solve the equation

$$-4x^2 + 4\left(16 - x^2\right) = 0 \Rightarrow -x^2 + \left(16 - x^2\right) = 0 \Rightarrow -2x^2 + 16 = 0 \Rightarrow x = \sqrt{8}$$

The dimenions are $x = \sqrt{8}, y = \sqrt{16 - x^2} = \sqrt{8}$

The maximum value of $A = 4xy$ is $A = 4\sqrt{8}\sqrt{8} = 32$.

HW-12. Using the steepest descent direction, find the minimum of $f(x_1, x_2) = 3x_1^2 + 2x_2$ starting at $x^{(0)} = [1 \quad 2]^T$ with a step size of $\alpha = 0.5$. Use 2 iterations.

Answer: $f(\boldsymbol{x}^{(0)}) = 3x_1^2 + 2x_2 = 3(1)^2 + 2(2) = 7$

From the analytical solution, the minimum point is at $x^* = [0 \quad 0]^T$ and $f(\boldsymbol{x}^*) = 0$.
Starting the process of iterations.

$$\boldsymbol{c} = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}^T \Rightarrow \boldsymbol{c}^{(0)} = \begin{bmatrix} 6x_1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6(1) \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\overline{\boldsymbol{c}} = \frac{\boldsymbol{c}}{\sqrt{\boldsymbol{c}^T\boldsymbol{c}}} \Rightarrow \overline{\boldsymbol{c}}^{(0)} = \frac{1}{\sqrt{6^2+2^2}} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.948683 \\ 0.316228 \end{bmatrix}$$

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} - 0.5\overline{\boldsymbol{c}}^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 0.5 \begin{bmatrix} 0.948683 \\ 0.316228 \end{bmatrix} = \begin{bmatrix} 0.525658 \\ 1.841886 \end{bmatrix}$$

$$f(\boldsymbol{x}^{(1)}) = 3x_1^2 + 2x_2 = 3(0.525658)^2 + 2(1.841886) = 4.466669$$

Next iteration

$$\boldsymbol{c}^{(1)} = \begin{bmatrix} 6x_1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6(0.525658) \\ 2 \end{bmatrix} = \begin{bmatrix} 3.15395 \\ 2 \end{bmatrix}$$

$$\overline{\boldsymbol{c}} = \frac{\boldsymbol{c}}{\sqrt{\boldsymbol{c}^T\boldsymbol{c}}} \Rightarrow \overline{\boldsymbol{c}}^{(1)} = \frac{1}{\sqrt{(3.15395)^2 + (2)^2}} \begin{bmatrix} 3.15395 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.844517 \\ 0.535529 \end{bmatrix}$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} - 0.5\overline{\boldsymbol{c}}^{(1)} = \begin{bmatrix} 0.525658 \\ 1.841886 \end{bmatrix} - 0.5 \begin{bmatrix} 0.844517 \\ 0.535529 \end{bmatrix} = \begin{bmatrix} 0.1034 \\ 1.574122 \end{bmatrix}$$

$$f(\boldsymbol{x}^{(2)}) = 3x_1^2 + 2x_2 = 3(0.1034)^2 + 2(1.574122) = 3.180317$$

HW-13. Apply Gauss-Newton method to $f(x_1, x_2) = \sum_{i=1}^{4} (2x_1x_2t_i - y_i)$ with data

$t = (2 \quad 3 \quad 6 \quad 7)^T$

$y = (3 \quad 6 \quad 8 \quad 9)^T$

using an initial guess $x = \begin{pmatrix} 2 \\ 0.5 \end{pmatrix}$

Answer:

Applying Gauss-Newton method to

$$f(x_1, x_2) = \sum_{i=1}^{4} (2x_1 x_2 t_i - y_i) \text{ with data}$$

$$t = [2 \quad 3 \quad 6 \quad 7]^T$$
$$y = [3 \quad 6 \quad 8 \quad 9]^T$$

Using an initial guess $x = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$

$$F(x) = (f_1(x) \quad f_2(x) \quad \ldots \quad f_m(x))^T$$

$$F(x) = \begin{bmatrix} 2x_1x_2t_1 - y_1 \\ 2x_1x_2t_2 - y_2 \\ 2x_1x_2t_3 - y_3 \\ 2x_1x_2t_4 - y_4 \end{bmatrix}$$

Evaluating $F(x)$ at the an initial guess $x = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$ and at $\begin{matrix} t = [2 \quad 3 \quad 6 \quad 7]^T \\ y = [3 \quad 6 \quad 8 \quad 9]^T \end{matrix}$

$$F(x) = \begin{bmatrix} 2x_1x_2t_1 - y_1 \\ 2x_1x_2t_2 - y_2 \\ 2x_1x_2t_3 - y_3 \\ 2x_1x_2t_4 - y_4 \end{bmatrix} = \begin{bmatrix} 2(2)(0.5)2 - 3 \\ 2(2)(0.5)3 - 6 \\ 2(2)(0.5)6 - 8 \\ 2(2)(0.5)7 - 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

$$\nabla F(x_1, x_2, \ldots, x_n)^T = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ . & . & . & . \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & . & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 2x_2t_1 & 2x_1t_1 \\ 2x_2t_2 & 2x_1t_2 \\ 2x_2t_3 & 2x_1t_3 \\ 2x_2t_4 & 2x_1t_4 \end{bmatrix} = \begin{bmatrix} 2(0.5)2 & 2(2)2 \\ 2(0.5)3 & 2(2)3 \\ 2(0.5)6 & 2(2)6 \\ 2(0.5)7 & 2(2)7 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 3 & 12 \\ 6 & 24 \\ 7 & 28 \end{bmatrix}$$

$$\nabla f(x) = \nabla F(x) F(x) = \begin{bmatrix} 2 & 3 & 6 & 7 \\ 8 & 12 & 24 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 61 \\ 244 \end{bmatrix}$$

For Gauss-Newton

$$\nabla F(x) \nabla F(x)^T p = -\nabla F(x) F(x)$$

$$\begin{bmatrix} 2 & 3 & 6 & 7 \\ 8 & 12 & 24 & 28 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 3 & 12 \\ 6 & 24 \\ 7 & 28 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -61 \\ -244 \end{bmatrix}$$

$\begin{bmatrix} 98 & 392 \\ 392 & 1568 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -61 \\ -244 \end{bmatrix}$, Using $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ it is found that the determinant $ad - bc = 0$. Thus, the two equations are dependent. Choose any $p_1 = 1$, from the the equation of the above matrix, $98 p_1 + 392 p_2 = -61 \Rightarrow p_2 = -\frac{159}{392} = -0.40561$

i.e., $p = \begin{bmatrix} 1 \\ -0.40561 \end{bmatrix}$. The new estimate of the solution is

$$x \to x + p = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ -0.40561 \end{bmatrix} = \begin{bmatrix} 3 \\ 0.09439 \end{bmatrix}$$

The iteration is continued.